Permutations with Up-Down Signatures of Nonnegative Partial Sums

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May 19-20, 2018
MIT PRIMES Conference

A Simpler Problem

Definition

Define a path p of length n as a sequence of points $p_0, p_1, \ldots, p_n$ in the plane such that $p_0 = (0, 0)$ and $p_i - p_{i-1} = (1, 1)$ or $(1, -1)$ for all positive integers $i \leq n$.

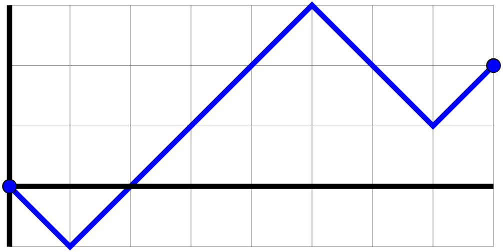

Figure: A path of length 8

Nonnegative Paths

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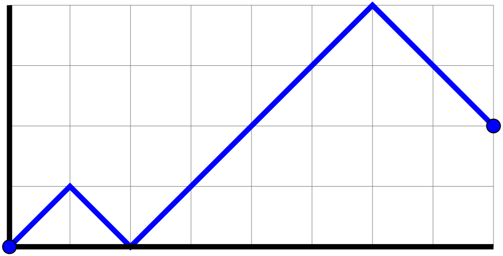
If each point p_i has nonnegative coordinates, then p is a nonnegative path.



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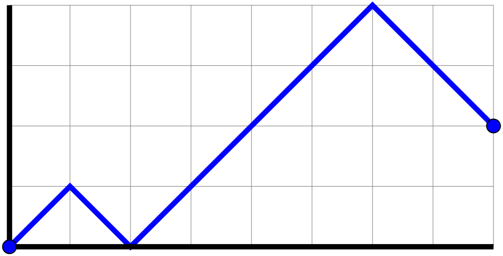

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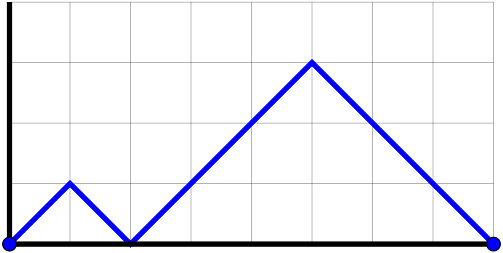

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There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ Dyck paths of length $2n$ (Chung and Feller, 1949).

Generalization to Permutations

Notation

Let $\mathfrak{S}_n$ denote the set of permutations of the numbers $1, 2, \ldots, n$. And for a permutation $w \in \mathfrak{S}_n$, let $w_1, w_2, \ldots, w_n$ be the entries in the permutation.

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Using $s(w)$, each permutation of $\mathfrak{S}_n$ maps to a path p_w of length $n-1$.

Example

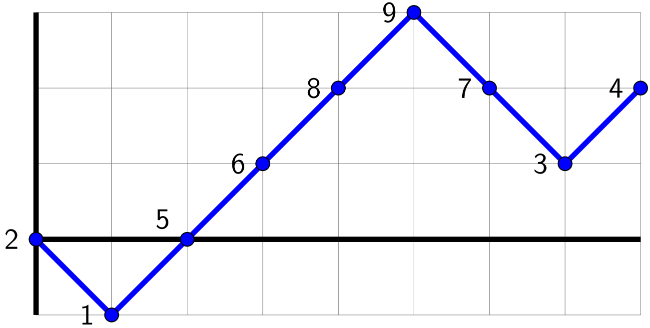

Figure: p_w for $w = 215689734$; $s(w) = (-1, +1, +1, +1, +1, -1, -1, +1)$

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$2n+1$	1	3	5	7	9
$A(2n+1,n)$	1	4	66	2416	156190
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- Our main goal is counting the number of nonnegative permutations

Conjecture and Examples

Conjecture (Callan, 2006)

The number of nonnegative permutations of $\mathfrak{S}_n$ *is* $(n-1)!!^2$ *if* n *is even, and* $n!!(n-2)!!$ *if* n *is odd.*

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When $n = 3$, there are exactly $3!!1!! = 3$ nonnegative permutations: $132, 123, 231$.

When $n = 4$, there are exactly $3!!^2 = 9$ nonnegative permutations: $1234, 1243, 1324, 1342, 1423, 2314, 2341, 2413, 3412$.

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2. $f(1) = 1$, $f(2n) = (2n-1)f(2n-1)$, and $f(2n+1) = (2n+1)f(2n)$.

Easy Cases

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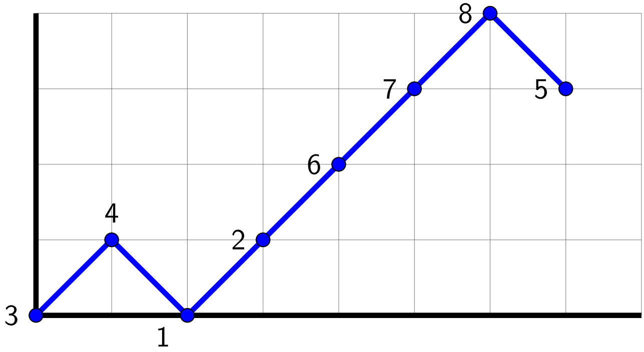
- For a $w \in F_{2n}$ add any number a from $\{1, 2, \ldots, 2n+1\}$ to w, and increment all numbers in w that are $\geq a$.
- For example, if $w = 1234$ and $a = 3$, then we obtain

$$(1234, 3) \mapsto 12453.$$

Example from F_{2n} to F_{2n+1}



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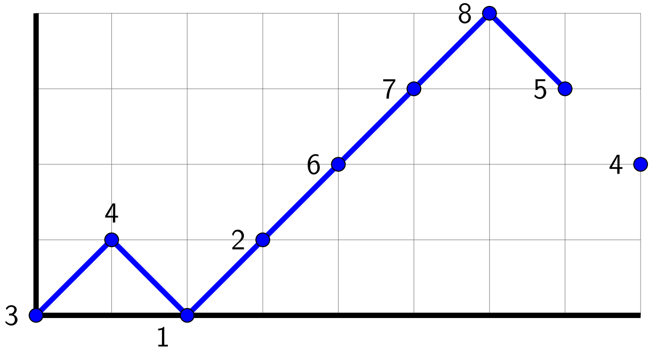

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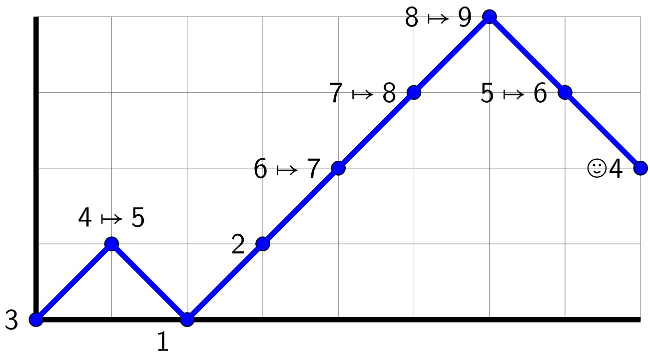

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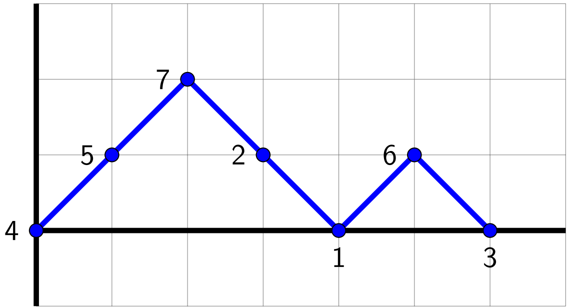
Note that before adding the 4, the path ended at $y = 2c + 1$.

Example that Fails from F_{2n-1} to F_{2n}



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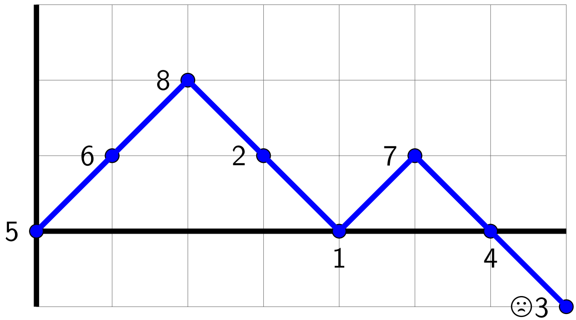

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$$(2n)f(2n-1) - \#\text{bad permutations} = f(2n)$$

- These "bad permutations" of length $2n$ begin with a permutation that is Dyck followed by a descent.
- If $D_{n,k}$ is the set of Dyck permutations of $\mathfrak{S}_n$ that end in k, then

$$\#\text{bad permutations of length } 2n = \sum_k k|D_{2n-1,k}|.$$

We would like to show that this equals $f(2n-1)$.

Possible Approaches

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- Interpret $\sum_k k|D_{2n-1,k}|$ as an expectation and use results derived from Szpiro and Shevelev.
- Consider the problem in a physics context: the numbers $f(n)$ appear in the analysis of spin-glass models and the Ising model.

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- My parents

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